



Fracture Mechanics

Soheil Mohammadi
School of Civil Engineering
University of Tehran

Tel: +98-21-6111 2258, Fax: +98-21-6640 3808, Cell: 0912-5030 392

Email: smoham@ut.ac.ir, Homepage: <http://chpc.ut.ac.ir/smohammadi>
Facebook: [prof.soheil.mohammadi](https://www.facebook.com/prof.soheil.mohammadi)



Classical Solutions for K and G

Classical Solutions for K and G

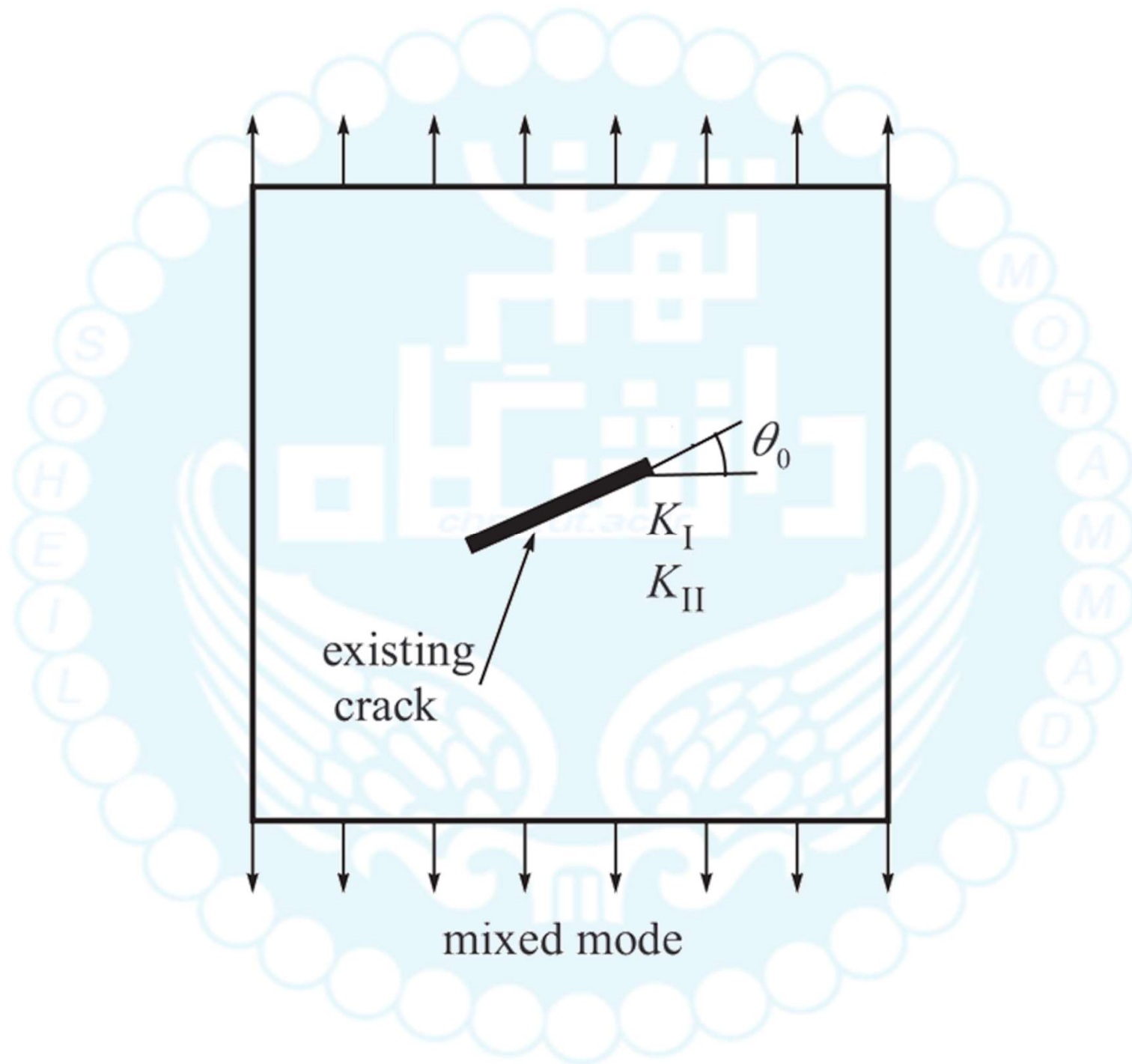
For most practical problems no analytical solution is available, and numerical techniques such as the finite element method, boundary element method, and more recently the meshless method should be used.

In this section, a review of available classical solutions mainly based on the finite element method is provided. The finite element method is mainly used as a powerful tool for the basic solution of displacement and stress states of the problem. In some cases, further information from the finite element solution may also be adopted.

The methods include:

1. Classical methods using the finite element method solely as a continuum based analytical tool.
2. Techniques in which the SIFs are directly evaluated as part of the global stiffness matrix.
3. Techniques through which the SIF can be computed from a standard finite element analysis via a special purpose post-processing technique.
4. Methods in which the singularity of the stress field at the crack tip is modelled.

No advanced finite element solutions or concepts are involved here. They will be comprehensively discussed in the second part of the course.



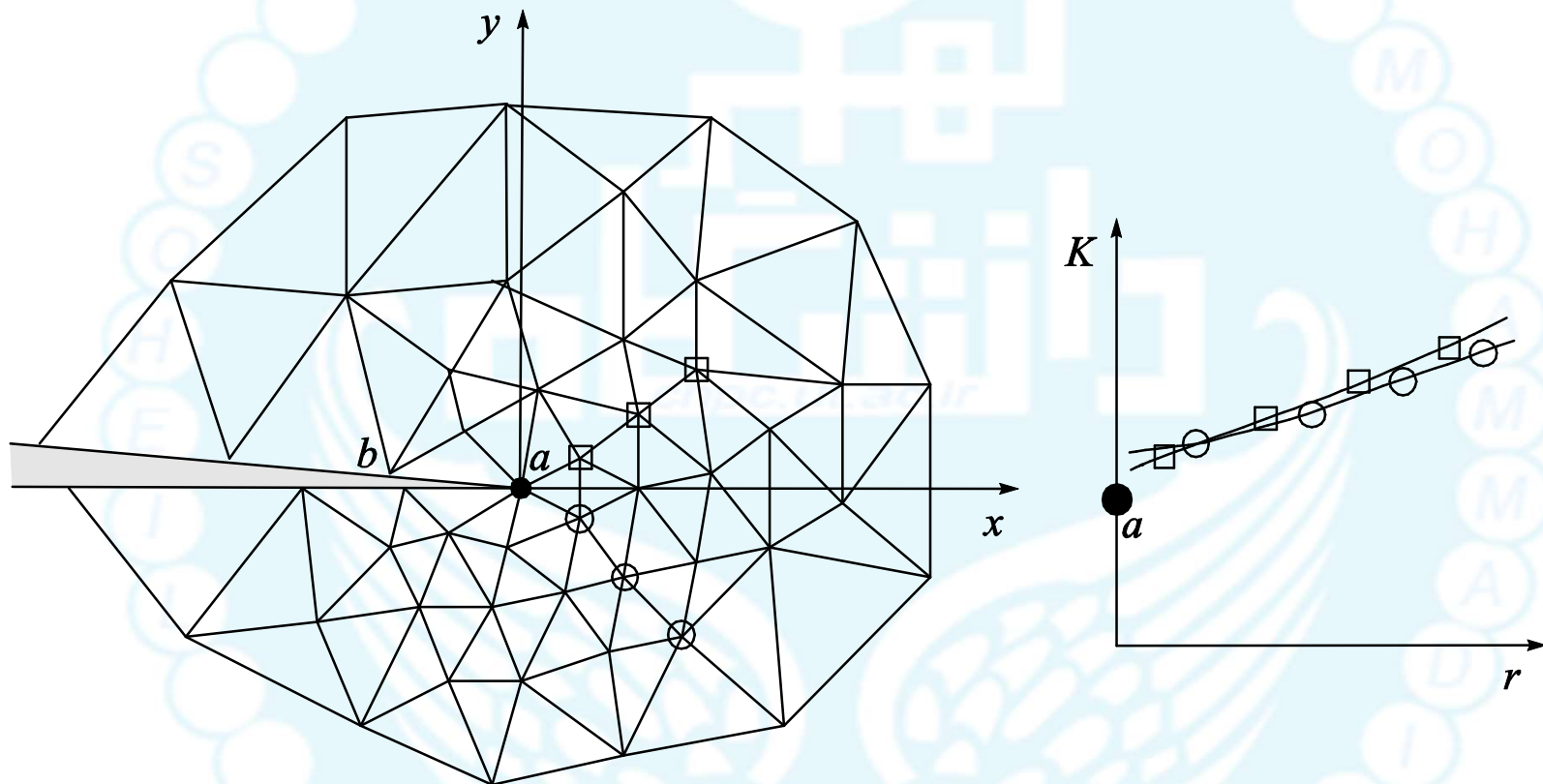
Displacement extrapolation/correlation method

The stress field at a crack tip is singular and conventional finite elements cannot represent it no matter how fine they are. Nevertheless, it was recognized in earlier simulations of LEFM that a very fine mesh is required at the crack tip if it is to be used to approximate the stress field for evaluation of the stress intensity factor.

According to Westergaard and Irwin solutions, displacement and stress states at any point near a crack tip can be determined if the values of corresponding stress intensity factors are known.

Alternatively, having known the values of displacement and stress states at any point, may allow for evaluation of stress intensity factors by equating the numerically obtained displacements with their analytical expressions in terms of SIFs.

Displacement extrapolation/correlation method



According to a linear elastic analysis:

$$\begin{aligned} \text{mode I: } & \begin{cases} 4G\sqrt{\frac{2\pi}{r}}u_x = K_{\text{I}} \left[(2\kappa - 1) \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \right] \\ 4G\sqrt{\frac{2\pi}{r}}u_y = K_{\text{I}} \left[(2\kappa + 1) \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right] \end{cases} \\ \text{mode II: } & \begin{cases} 4G\sqrt{\frac{2\pi}{r}}u_x = K_{\text{II}} \left[-(2\kappa + 3) \sin \frac{\theta}{2} - \sin \frac{3\theta}{2} \right] \\ 4G\sqrt{\frac{2\pi}{r}}u_y = K_{\text{II}} \left[(2\kappa - 3) \cos \frac{\theta}{2} + \cos \frac{3\theta}{2} \right] \end{cases} \end{aligned}$$

The stress intensity factors can be determined at different radial distances from the crack tip by equating the numerically (FEM) obtained displacements with their analytical expression in terms of the SIF.

The computed SIFs are plotted with respect to their distance r from the crack tip.

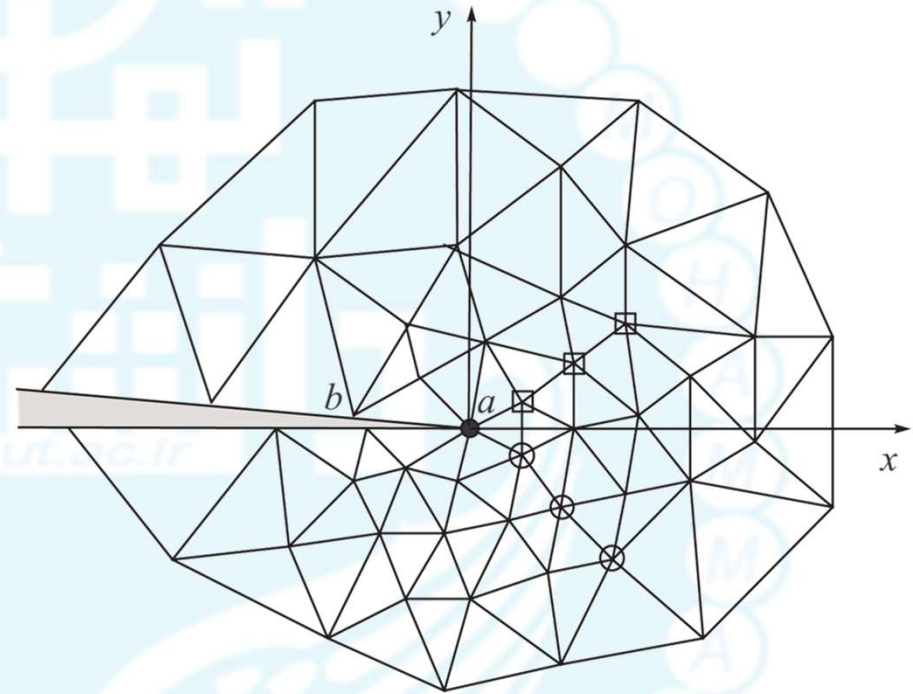
Clearly, different SIFs may be obtained from different radial paths and even different points on a same path.

A line/curve regression is performed on the plot and then extrapolated to $x=0$ to obtain the best estimate of SIF.

Usually, a set of points b (with θ equal to π) are considered with respect to the crack tip a .

$$K_I = \mu \sqrt{\frac{2\pi}{r}} \frac{u_y^b - u_y^a}{\kappa + 1}$$

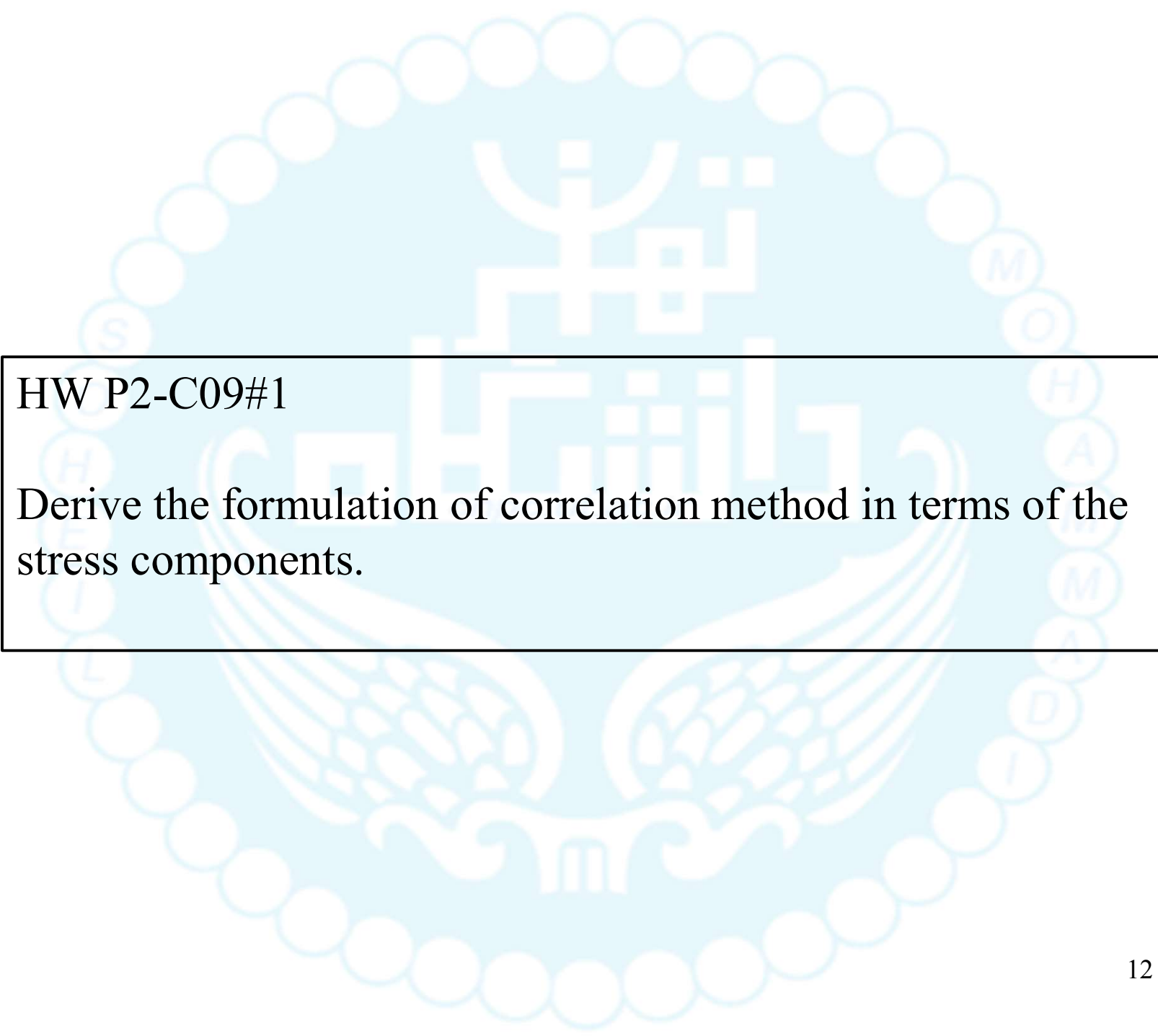
$$K_{II} = \mu \sqrt{\frac{2\pi}{r}} \frac{u_x^b - u_x^a}{\kappa + 1}$$



Note that u^a represents the total crack tip displacement (obtained from the numerical analysis) and is different from the local asymptotic displacement at the crack tip.

Displacement extrapolation:

- This technique was the predominant one prior to the development of the quarter-point singular elements.
- The method is very sensitive to selection of points.
- Acceptable results may only be obtained in very fine mesh and points close to the crack tip are used.
- The same procedure for stresses can also be used, although it is likely to yield less accurate predictions due to singular nature of stress state near the crack tip and to far less accurate FEM solution for stress near the crack tip.
- Due to its extreme simplicity, the method is still being used for many problems, especially as a first estimate for new complex fracture problems.



HW P2-C09#1

Derive the formulation of correlation method in terms of the stress components.

Mode I energy release rate

A direct method for evaluation of the mode I stress intensity factor and the energy release rate is based on the direct definition of G ,

$$G = \frac{K_I^2}{E'} = -\frac{\partial \Pi}{\partial a} \cong -\frac{\Delta \Pi}{\Delta a}$$

and finding the difference in the total strain energy of the system for initial and (virtually) extended crack lengths, respectively:

$$\begin{array}{ll} a & \rightarrow \Pi \\ a + \Delta a & \rightarrow \Pi + \Delta \Pi \end{array}$$

Therefore, the algorithm requires **two completely separate analyses**, which is a computationally very expensive task.

The strain energy can be determined:

- in terms of the nodal displacement $\mathbf{u}$ and the global stiffness matrix $\mathbf{K}$,

$$\Pi = \mathbf{u}^T \mathbf{K} \mathbf{u}$$

- in terms of the external load $\mathbf{P}$ and displacements $\mathbf{u}$.

$$\Pi = \mathbf{u}^T \mathbf{P}$$

It should be noted that while the term G is general and does not depend only to mode I, this method is only directly used for evaluation of mode I stress intensity factor G .

The virtual extended crack length Δa is always assumed along the crack length to fully represent the mode I.

Mode I stiffness derivative Virtual crack model

A major problem with the previous method is the requirement for two complete analyses for the evaluation of G .

It should be noted, however, that the stiffness matrix of the second analysis (associated with $a + \Delta a$) is only slightly altered from the first one (associated with a). A remedy, therefore, is to use relaxation methods to reduce the computational costs for the second analysis, as independently proposed by Parks (1974) and Hellen and Blackburn (1975), who called the method **stiffness derivative** and **virtual crack** model, respectively.

In general,

$$\Pi = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{P}$$

$\mathbf{K}$ = stiffness matrix, $\mathbf{P}$ = force vector, $\mathbf{u}$ = displacement vector

$$\begin{aligned} G &= -\frac{\partial \Pi}{\partial a} = -\frac{\partial}{\partial a} \left[\frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{P} \right] \\ &= -\left\{ \left[2 \times \frac{1}{2} \frac{\partial \mathbf{u}^T}{\partial a} \mathbf{K} \mathbf{u} + \frac{1}{2} \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial a} \mathbf{K} \mathbf{u} \right] - \left[\frac{\partial \mathbf{u}^T}{\partial a} \mathbf{P} + \mathbf{u}^T \frac{\partial \mathbf{P}}{\partial a} \right] \right\} \\ &= -\left\{ \frac{\partial \mathbf{u}^T}{\partial a} \underbrace{(\mathbf{K} \mathbf{u} - \mathbf{P})}_0 + \frac{1}{2} \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial a} \mathbf{K} \mathbf{u} - \mathbf{u}^T \frac{\partial \mathbf{P}}{\partial a} \right\} \\ &= -\left\{ \frac{1}{2} \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial a} \mathbf{K} \mathbf{u} - \mathbf{u}^T \frac{\partial \mathbf{P}}{\partial a} \right\} \end{aligned}$$

Assuming that the loading is unaltered during the virtual crack extension,

$$G = -\frac{\partial \Pi}{\partial a} = -\left\{ \frac{1}{2} \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial a} \mathbf{u} - \underbrace{\mathbf{u}^T \frac{\partial \mathbf{P}}{\partial a}}_0 \right\} = -\frac{1}{2} \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial a} \mathbf{u}$$

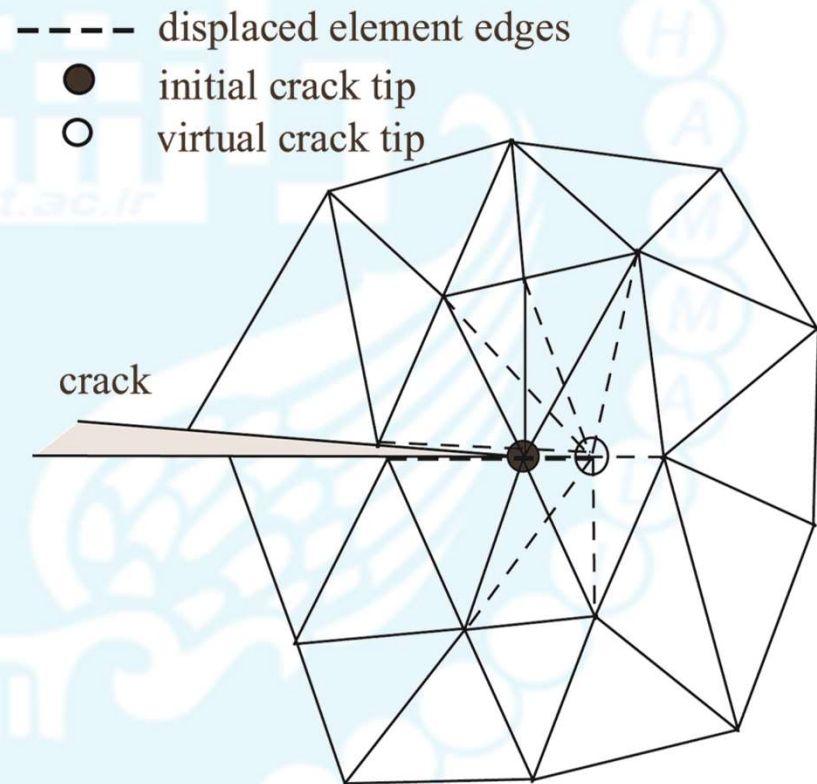
which can be also assumed as unaltered displacement during the virtual crack extension,

$$G = -\frac{\partial \Pi}{\partial a} = -\frac{\partial}{\partial a} \left[\frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{P} \right] \xrightarrow{\frac{\partial \mathbf{u}}{\partial a} = 0} = -\frac{1}{2} \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial a} \mathbf{u}$$

Therefore, the derivative of the stiffness matrix is required for evaluation of G . Instead of cumbersome numerical differentiations, this is usually computed by perturbing the elements around the crack tip and evaluating the modified part of the stiffness matrix associated with the nodes of elements along the crack extension.

$$1 - \begin{cases} a & \rightarrow \mathbf{u}, \mathbf{K} \\ a + \Delta a & \rightarrow \Delta \mathbf{K} \end{cases}$$

$$2 - G = -\frac{1}{2} \mathbf{u}^T \frac{\Delta \mathbf{K}}{\Delta a} \mathbf{u}$$

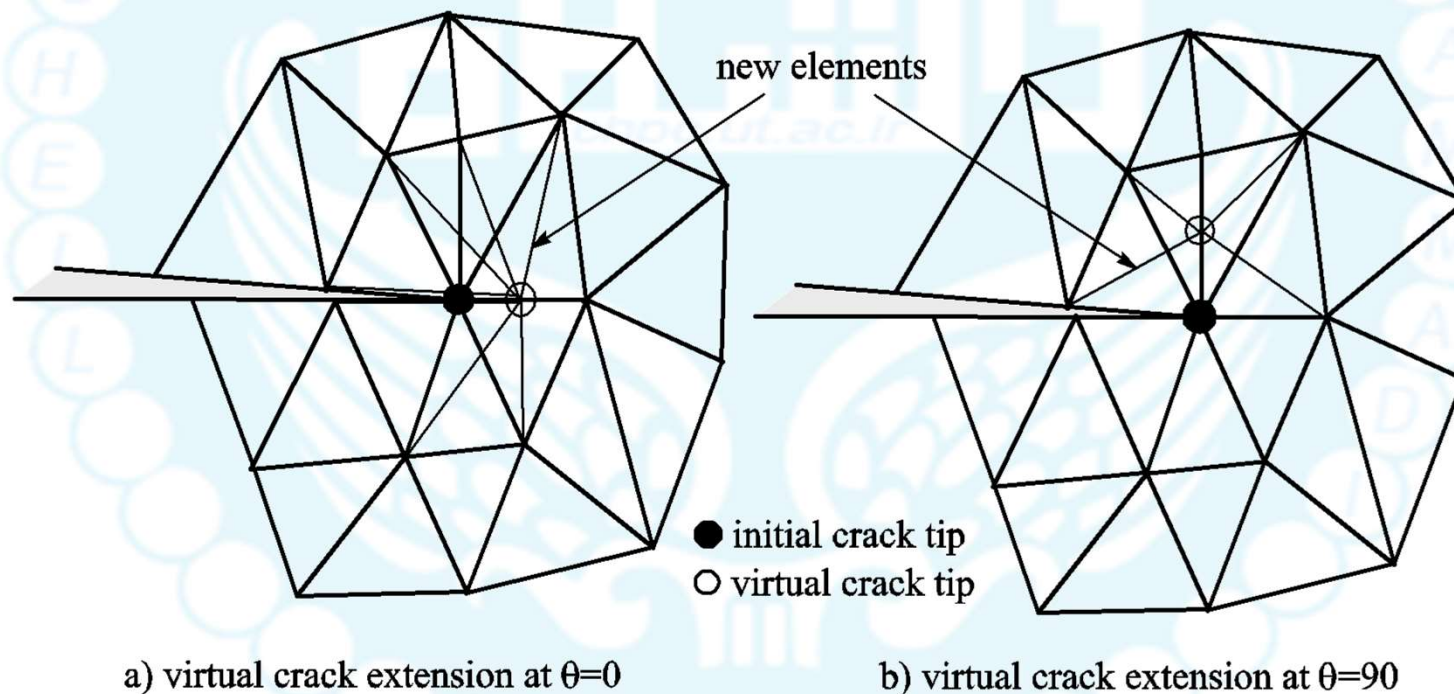


Mode I stiffness derivative/Virtual crack model:

- The method was used by computational community in late 90's.
- The virtual crack extension should be performed with a move of crack tip node, while the mesh connectivity remains unchanged. As a result, post-processing calculation of changes of stiffness can only be performed in a few number of elements. Therefore, the method is much faster than the method of mode-I energy release rate.
- The method requires FEM expertise (to some extent) and access to the source code of FEM in order to evaluate stiffness derivative.
- Better results are obtained by singular finite elements.

Two virtual crack extensions for mixed mode cases

A natural extension to the virtual crack model to include effects of mixed mode fracture is to use the two virtual crack extensions model.



A rather simple case with a closed form solution was proposed by Hellen and Blackburn (1975) based on the following expressions of the energy release rates for virtual crack extensions at with $\theta=0$ and $\theta=\pi/2$:

$$G_1 = \frac{K_I^2 + K_{II}^2}{E'}$$

$$G_2 = \frac{-2K_I K_{II}}{E'}$$

It should be noted that G is defined for the whole fracture problem, and definitions of G_1 and G_2 are not related to mode I and mode II components of G .

The procedure can be started by computing the total strain energy U_s for an initial crack length a . Then the crack length is extended to $a + \Delta a$ along $\theta=0$ and $\theta=\pi/2$ to determine G_1 and G_2 , respectively. Computing the values of G requires one of previous methods.

Solving the simultaneous equations lead to computation of the stress intensity factors:

$$K_I, K_{II} = \frac{1}{2} \left(s \pm \sqrt{s^2 + \frac{8G_2}{\kappa'}} \right)$$

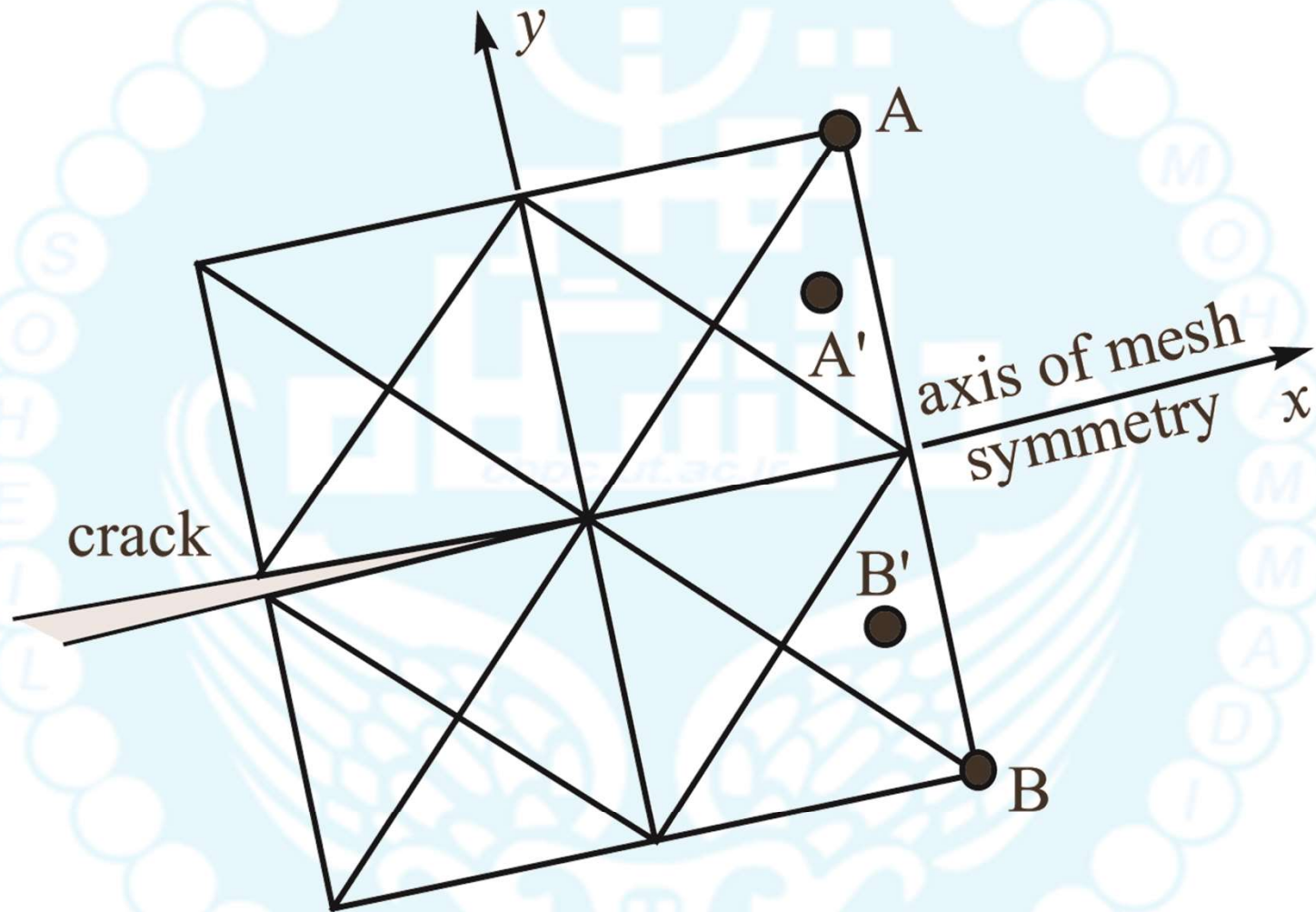
$$s = 2 \sqrt{\frac{G_1 - G_2}{\kappa'}}$$

$$\kappa' = \frac{(1+\nu)(1+\kappa)}{E}$$

Displacement decomposition based on Single virtual crack extension

Previous techniques were computationally expensive and inefficient because they required at least one complete finite element analysis, followed by two separate analyses or two virtual crack extensions.

The present approach, proposed by Ishikawa (1980) and Sha (1984), requires only one analysis (or one virtual crack extension) but it is limited to symmetrical local elements around the crack tip.

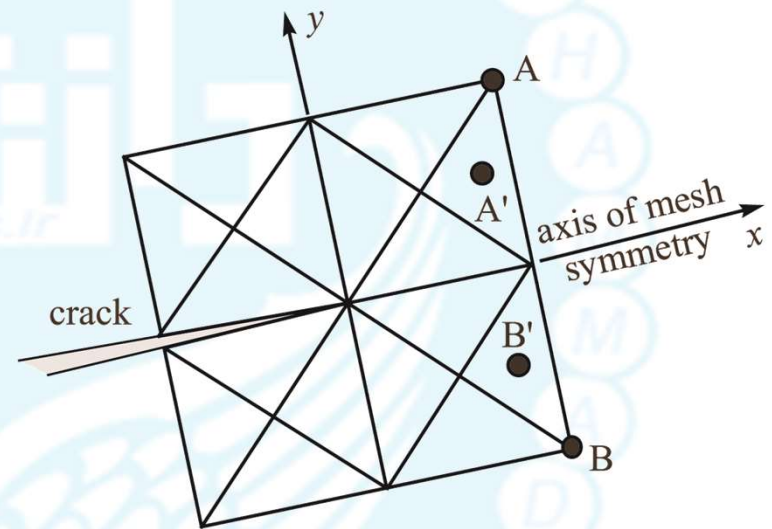


The nodal displacements for a symmetrical local mesh around the crack tip can be decomposed into two local symmetric U^I and anti-symmetric U^{II} components about the crack plane:

$$U = U^I + U^{II}$$

$$U^I = \begin{Bmatrix} u_x^I \\ u_y^I \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} u_x^A + u_x^B \\ u_y^A - u_y^B \end{Bmatrix}$$

$$U^{II} = \begin{Bmatrix} u_x^{II} \\ u_y^{II} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} u_x^A - u_x^B \\ u_y^A + u_y^B \end{Bmatrix}$$



where (x, y) are the local crack tip coordinate system. Then, the local mode I (U^I) and mode II (U^{II}) components can be used directly to determine K_I and K_{II} , respectively.

This method decouples the mixed model into pure mode I and mode II states. Each state will need one of the previous methods to compute the corresponding K or G .

Similar rules can be used to decompose the stress field into mode I and mode II stress components for evaluation of the energy release rate G (Nikishkof and Atluri, 1987).

It should be noted that this method can be used with any two symmetric points with respect to the crack direction, even on any arbitrarily generated mesh (such as A' and B' in the figure).

HW P2-C09#2

Prove that in the displacement decomposition method, U^I and U^{II} represent pure mode I and mode II, respectively.

